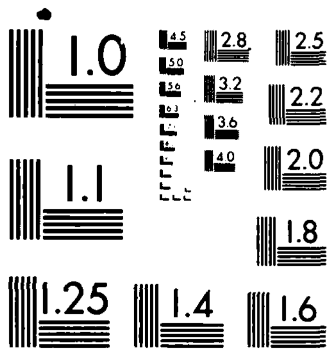
AD-A161 043 MARRIAGE AND MONEY(U) STANFORD UNIV CA INST FOR MATHEMATICAL STUDIES IN THE SOCIAL SCIENCES T QUINT AUG 85 TR-472 N00014-79-C-0685 1/1

UNCLASSIFIED F/G 12/1 NL

END

FILMED

DTIC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

TECHNICAL REPORT SERIES

AD-A161 043

MARRIAGE AND MONEY

by

Thomas Quint

Technical Report No. 472

August 1985

THE ECONOMICS SERIES

INSTITUTE FOR MATHEMATICAL STUDIES IN THE SOCIAL SCIENCES
FOURTH FLOOR, ENCINA HALL
STANFORD UNIVERSITY
STANFORD, CALIFORNIA

85 10 15 130

1

MARRIAGE AND MONEY

by

Thomas Quint

Technical Report No. 472

August 1985

PREPARED UNDER
OFFICE OF NAVAL RESEARCH GRANT ONR-N00014-79-C-0685

THE ECONOMICS SERIES

INSTITUTE FOR MATHEMATICAL STUDIES IN THE SOCIAL SCIENCES
Fourth Floor, Encina Hall
Stanford University
Stanford, California
94305

MARRIAGE AND MONEY*

by

Thomas Quint

1. Introduction

Much of the economic and game theoretic literature focuses on the questions of existence and characterization of equilibrium points or core points. Of equal interest are schemes which actually calculate such values. Indeed, predicting the behavior of economic markets becomes a lot easier in the presence of such algorithms. In this paper, we tackle this issue for the "marriage problem" with transferable utility. Such matching problems are useful models of, for example, the market in which jobs are to be matched to prospective employees.

Gale and Shapley [1962] were the first to formally pose a "marriage problem". Their setup was as follows. Consider a system with two types of agents, hereafter called men and women. Each man has a preference ordering over the women; likewise, each woman ranks the men. The objective is to find a "stable" matching; i.e., one in which no unmarried couple will willingly leave their spouses and run off together. To solve the problem, they presented the well-known "Gale-Shapley" algorithm. Each man begins by proposing to his favorite woman. If no woman receives more than one offer, the matching so defined is stable; if not, any woman with two or more offers rejects all but the most appealing. The rejected men then propose to their second choices, and so on. The process ends when no woman has more than one pending offer. At this

*This research was supported by the Office of Naval Research Grant ONR-N00014-79-C-0685 at the Institute for Mathematical Studies in the Social Sciences, Stanford University. I would like at this time to thank my dissertation advisor, Professor Kenneth Arrow, for his support and input in this research.

point, not only is the implied matching stable, but it is "man optimal" as well, i.e., it is the stable outcome which is "best" for the men. Interestingly, this method closely resembled the algorithm long used by the medical profession in its assignment of interns to residency programs [Roth, 1983].

In 1984 Gale and Demange proposed a major innovation into this basic model. The new wrinkle was that monetary transfer was now allowed between marriage partners. Thus, each man no longer expresses his utility for prospective mates via a simple preference ordering. Instead, he now has a series of utility functions, one for each woman, each with the amount of monetary transfer as the independent variable. In addition, each man has a "reservation utility"--that is, a utility for being left unmatched. Of course, women's utilities for men have the same structure.

This, then, is our model. Note that it is now much more complex than the "simple" Gale-Shapley setup, because now, in our search for a stable outcome, we must not only specify a matching, but also monetary transfers to take place within each couple.

The paper is organized as follows. Section 2 defines the problem and gives a proof of the existence of a stable solution. Section 3 discusses the "linear separable" case; i.e., where, in addition to the linearity of the utility functions, we assume that a person's utility for money does not depend on with whom he/she is matched. Section 4 covers the "linear nonseparable" case, where we drop the latter assumption. Unlike the previous case, we do not present a method for finding all stable solutions. However, we do present an extension of the Gale-Shapley algorithm which always calculates a stable solution, and always generates a "man-optimal" matching. Next, in Section 5 we generalize the algorithm presented in Section 4 to cover nonlinear utility functions as well. Finally, Section 6 suggests some areas for further research.

2. Description of Model

The data needed to describe the model is as follows. Set

m = number of men in the model

n = number of women in the model

$u_{ij}(\alpha)$ = utility to man i of being matched to woman j and receiving payment of α

$v_{ij}(\alpha)$ = utility to woman j of being matched to man i and receiving payment of α

R_i = reservation utility of man i

S_j = reservation utility of woman j

Assume that both $u_{ij}(\alpha)$ and $v_{ij}(\alpha)$ are strictly increasing in α.

Definition: A matching is a set of man-woman pairs $\{(M_i,W_j)\}$ with the property that no person is a member of more than one pair.

Definition: A feasible payoff is a set of utility levels $\{u_i\}$ and $\{v_j\}$ such that there exists a matching μ with payments $\{\alpha_i\}$ to the matched men satisfying:

(2.1)

1) If i is not matched by μ, $u_i = R_i$.
   If j is not matched by μ, $v_j = S_j$.

2) If i is matched by μ, and α_i is the amount man i receives from his mate $\mu[i]$, then $u_i \leq u_{i\mu[i]}(\alpha_i)$ and $v_{\mu[i]} \leq v_{i\mu[i]}(-\alpha_i)$.

In other words, a feasible distribution of utility is one which the system has the resources to effect. Note that we can also write condition 2) as:

$$f_{i\mu[i]}(u_i) + g_{i\mu[i]}(v_j) \leq 0 \ , \tag{2.2}$$

where f_{ij} and g_{ij} are the inverse functions of u_{ij} and v_{ij}.

<u>Definition</u>: A <u>stable</u> payoff is a set of utility levels $\{u_i\}$ and $\{v_j\}$ such that:

(2.3)

1) No single person wishes to run off, i.e., $u_i \geq R_i$ and $v_j \geq S_j$ for every man i and woman j.

2) No "unmarried" couple wishes to run off, i.e., there are no α, i, and j for which $u_{ij}(\alpha) > u_i$ and $v_{ij}(-\alpha) > v_j$. Again, we can equivalently write:

$$f_{ij}(u_i) + g_{ij}(v_j) \geq 0 \quad \text{for every } i,j. \tag{2.4}$$

<u>Remark</u>: Note that monetary transfers are only allowed between a man and his prospective marriage partner.

Remark: Note that by (2.2) and (2.4), any stable feasible payoff satisfies

(2.5) $$f_{ij}(u_i) + g_{ij}(v_j) = 0 \quad \text{for } i \text{ matched to } j.$$

This complementarity surfaces later [as the constraint "$s_{ij}x_{ij} = 0$" in Sections 4) and 5)].

Definition: Define the core as the set of all stable feasible payoffs. Define a core matching as any matching μ under which stable feasible payoffs exist. Let CM be the set of all core matchings.

Next, we state a result of Demange and Gale [1984]:

Lemma 2.1: The set of men's utilities $\{u_i\}$ for which there exist utilities $\{v_j\}$ satisfying

$$(\{u_i\}, \{v_j\}) \text{ is in the core}$$

is a lattice.

Proof: See Demange and Gale (1984).

Thus, it makes sense to define the man-optimal utilities as that vector in the core which achieves the highest utility levels for the men. Also, if μ is a core matching under which the man-optimal utilities are feasible, we call it a man optimal matching.

<u>Example</u>: Suppose we have:

	u_{ij} v_{ij}	women $j=1$	 $j=2$	 $j=3$	R_i
m	$i=1$	α $2\alpha+6$	$\alpha+1$ $\alpha+3$	$2\alpha+2$ $\alpha+2$	0
e	$i=2$	$5\alpha-4$ $5\alpha+8$	$\alpha+1$ $2\alpha+8$	$3\alpha-5$ $6\alpha-2$	-1
n	$i=3$	$2\alpha+3$ $\alpha-6$	α $\alpha+2$	$2\alpha+1$ $\alpha-1$	1
	S_j	0	2	2	

Consider the matching $\mu_1 = \{(1,3),(2,1)\}$. Then the utilities are:

$$(u_1,u_2,u_3) = (2 + 2\alpha_1, -4 + 5\alpha_2, \quad 1 \quad)$$
$$(v_1,v_2,v_3) = (8 - 5\alpha_2, \quad 2 \quad , 2 - \alpha_1)$$

In order for μ to be a core matching, we need $u_i \geq R_i$, $v_j \geq S_j$, i.e.,

$$-1 \leq \alpha_1 \leq 0 \quad \text{and} \quad 3/5 \leq \alpha_2 \leq 8/5 .$$

Finally we need to satisfy $f_{ij}(u_i) + g_{ij}(v_j) \geq 0$ for <u>every</u> (i,j) combination. But this holds if we set:

$$\alpha_1 = 0 \quad \text{and} \quad \alpha_2 = 8/5 .$$

<u>Remark</u>: It is important to note that more than one core matching can exist. For instance, one can verify that $\mu_2 = \{(1,1),(2,2)\}$ is also in CM. Interestingly, the man-optimal utilities $(u_1,u_2,u_3) = (3,4,1)$ can be attained with μ_2 but not with μ_1.

Theorem 2.1: The core of this game is nonempty.

Proof: Quinzii's (1984) proof in which she shows that certain types of "pairing" models are balanced games applies here. However, we supply a slightly different proof which is more specific to this particular game.

Again, we show that the game is balanced. So, let T be any balanced set of coalitions, and, for every $S \in T$, let δ_S be the "balancing weight". Also, let V^S be the set of utility vectors that the members of S can attain by themselves without outside intervention. Next, suppose $\mathbf{w} = (\{u_i\},\{v_j\})$ is a utility vector, with $\mathbf{w}^S \in V^S$ for every $S \in T$. We need to show that $\mathbf{w} \in V^N$, where N is the set of all men and women (see Scarf, 1967).

Claim: Without loss of generality, we can assume that T contains only 1-person coalitions and 1-man-1-woman coalitions.

Proof: Consider any coalition $S \in T$, with $\mathbf{w}^S \in V^S$. Let $\mu^S = \{(M_1,W_1),\ldots,(M_p,W_p)\}$ be a matching within S which attains $\mathbf{w}^S$ for the members of S, and let $\{M_{p+1},\ldots,M_{p+q},W_{p+1},\ldots,W_{p+r}\}$ be the members of S who are unmatched by μ^S. Then replace S in T by $(S_1,\ldots,S_{p+q+r})$, where:

$$
\begin{array}{lll}
S_t = (M_t,W_t) & \text{if} & t \leq p \\
S_t = (M_t) & \text{if} & p+1 \leq t \leq p+q \\
S_t = (W_{t-q}) & \text{if} & p+q+1 \leq t \leq p+q+r.
\end{array}
$$

Also, let

$$\delta_{S_t} = \delta_S \qquad \text{for all } t.$$

Repeat this process for every $S \in T$, finally obtaining a (much larger) set of coalitions T^*. T^* satisfies:

1) T^* is balanced.

2) T^* contains only 1-person coalitions and 1-man-1-woman coalitions.

3) $\mathbf{w}^S \in V^S$ for every $S \in T \iff \mathbf{w}^S \in V^S$ for every $S \in T^*$.

These three facts imply the Claim.

Assuming the Claim, we now prove $\mathbf{w} \in V^N$ by induction on the number of elements in T. It is obvious when $|T| = 1$. So assume $|T| = k$.

<u>Case 1</u>: $\delta_{S^*} = 0$ for some $S^* \in T$.
Then $T - S^*$ is balanced, and thus,

$$\mathbf{w}^S \in V^S \text{ for every } S \in T \implies \mathbf{w}^S \in V^S \text{ for every } S \in T - S^*$$
$$\implies \mathbf{w} \in V^N \text{ by the inductive hypothesis.}$$

<u>Case 2</u>: $\delta_{S^*} = 1$ for some $S^* \in T$.
Let S^{*c} be the complement of S^*, i.e., the set of men and women not contained in S^*. Then $T - S^*$ is a balanced set over S^{*c}. Thus,

$$\mathbf{w}^S \in V^S \text{ for every } S \in T \implies \mathbf{w}^S \in V^S \text{ for every } S \in T - S^*$$
$$\implies \mathbf{w}^{S^{*c}} \in V^{S^{*c}} \text{ by the inductive hypothesis.}$$

But this, combined with $\mathbf{w}^{S^*} \in V^{S^*}$ and the fact that the game is superadditive, imply $\mathbf{w} \in V^N$ as desired.

Remark: So we now assume that $0 < \delta_S < 1$ for every $S \in T$. This implies that every player in the game is a member of at least two coalitions.

Case 3: There is at least one coalition in T with only one person. Let A be the $(m + n) \times (k)$ "0-1"-matrix in which $A_{ij} = 1$ iff the i^{th} person is in the j^{th} coalition. Thus, T balanced means that

(2.6) $$A\underline{\delta} = \underline{e},\ \underline{\delta} \geq 0$$

has a solution, where $\underline{e}$ is a vector of 1's. Next, set:

C = the number of columns in A

R = the number of rows in A

$\bar{0}$ = the number of "1's" in A

By the Claim and our assumption in Case 3, $\bar{0} < 2C$. And, by the last Remark, $\bar{0} \geq 2R$. Thus, $R < C$. But this in conjunction with (2.6) implies:

(2.7) $$A\underline{\delta} = \underline{e},\ \underline{\delta} \geq 0,\ \delta_S = 0 \text{ for some } S$$

has a solution (see Gale, 1960). So, by Case 1, $\mathbf{w}^N \in V^N$.

<u>Case 4</u>: T consists of only married couples.

Now $\bar{0} = 2C$ and $\bar{0} \geq 2R$, but if $\bar{0} > 2R$, then $R < C$ again and we can use the argument from Case 3 to show $\mathbf{w} \in V^N$. So assume that $\bar{0} = 2R = 2C$. Then we have that:

1) Every man is in coalitions with exactly 2 women.

2) Every woman is in coalitions with exactly 2 men.

3) # of men in T = # of women in T = k.

So, without loss of generality, suppose $\{(M_1,W_1),(M_1,W_2),(M_2,W_1)\} \subset T$. Then $\delta_{(M_1,W_2)} = \delta_{(M_2,W_1)} = 1 - \delta_{(M_1,W_1)}$.

<u>Subcase A</u>: $(M_2,W_2) \in T$.

Then $\delta_{(M_2,W_2)} = \delta_{(M_1,W_1)}$, and we can replace $\delta_{(M_1,W_1)},\delta_{(M_1,W_2)},\delta_{(M_2,W_1)},\delta_{(M_2,W_2)}\}$ by $\{1,0,0,1\}$ and these are still balancing weights. But by Case 1, $\mathbf{w} \in V^N$.

<u>Subcase B</u>: $(M_2,W_2) \notin T$.

Then, without loss of generality, suppose $\{(M_2,W_3),(M_3,W_2)\} \subset T$.

i) $(M_3,W_3) \in T$. In this case we can again replace

$$\{\delta_{(M_1,W_1)},\delta_{(M_1,W_2)},\delta_{(M_2,W_1)},\delta_{(M_2,W_3)},\delta_{(M_3,W_2)},\delta_{(M_3,W_3)}\}$$

by $\{1,0,0,1,1,0\}$ and still have balancing weights. So $\mathbf{w} \in V^N$ by Case 1.

ii) $(M_3,W_3) \notin T$. Then assume $\{(M_3,W_4),(M_4,W_3)\} \subset T\ldots$

...

Continuing in this fashion, we must eventually reach a $kl \leq k$ for which $(M_{kl},W_{kl}) \subset T$. And we can replace

$$\{\delta_{(M_1,W_1)},\delta_{(M_1,W_2)},\delta_{(M_2,W_1)},\ldots,\delta_{(M_{kl},W_{kl-1})},\delta_{(M_{kl},W_{kl})}\}$$

by $\{1,0,0,1,1,0,0,\ldots,1,1,0\}$ if kl is odd, and

$\{1,0,0,1,1,0,0,\ldots,0,0,1\}$ if kl is even,

and these are again balancing weights. So again we can use Case 1.

3. The Linear Separable Case

Now that we know core matchings with stable feasible payoffs always exist, we consider the problem of trying to calculate them. We first examine the linear separable case, where we allow the following assumptions:

1) $u_{ij}(\alpha)$ and $v_{ij}(\alpha)$ are linear for every i and j.
2) A person's utility for money does not depend on with whom he/she is matched.

After the appropriate normalization, this is equivalent to assuming that the utility functions have the form

$$u_{ij}(\alpha) = u_{ij} + \alpha \quad \text{and} \quad v_{ij}(\alpha) = v_{ij} + \alpha,$$

where u_{ij} and v_{ij} are constants. Let $c_{ij} = u_{ij} + v_{ij}$.

With the problem so defined, note that it is a slight generalization of Shapley and Shubik's housing market [1972]. In that model, the buyers and sellers correspc ɟ to our men and women. However, sellers only care about the monetary transfer they receive and not who it comes from. In addition, Shapley and Shubik assume that all reservation utilities are zero.

However, their formulation of the problem as a linear program and the interpretation of the dual variables as utilities remains valid. To wit:

<u>Theorem 3.1</u>: μ is a core matching iff it maximizes <u>global utility</u>, i.e, it maximizes:

$$\sum_{(i,j)\epsilon\mu} c_{ij} + \sum_{\substack{i \text{ not} \\ \text{matched}}} R_i + \sum_{\substack{j \text{ not} \\ \text{matched}}} S_j \tag{3.1}$$

<u>Proof</u>: Consider the linear program (P1):

$$\max \sum_{i,j=1}^{m,n} c_{ij}p_{ij} + \sum_{i=1}^{m} R_i q_i + \sum_{j=1}^{n} S_j r_j$$

$$\text{s.t.} \quad \sum_{j=1}^{n} p_{ij} + q_i = 1 \tag{3.2}$$

$$\sum_{i=1}^{m} p_{ij} + r_j = 1 \tag{3.3}$$

$$p_{ij},\ q_i,\ r_j \geq 0\ . \tag{3.4}$$

This is of course equivalent to the program (P2):

$$\max \; [\sum_{i=1}^{m} R_i + \sum_{j=1}^{n} S_j \;] \; + \sum_{i,j=1}^{m,n} c_{ij}p_{ij} - \sum_{i=1}^{m} R_i \sum_{j=1}^{n} p_{ij} - \sum_{j=1}^{n} S_j \sum_{i=1}^{m} p_{ij}$$

(3.5) $$\text{s.t.} \quad \sum_{j=1}^{n} p_{ij} \leq 1$$

(3.6) $$\sum_{i=1}^{m} p_{ij} \leq 1$$

(3.7) $$p_{ij} \geq 0 \; .$$

Let $\{\bar{p}_{ij}\}$ solve (P1) or (P2).

<u>Claim</u>: Maximizing (3.1) is equivalent to solving program (P1).

<u>Proof</u>: It is clear that the Claim follows if we can show that program (P1) can always be solved with p_{ij}'s, q_i's, and r_j's all equal to 0 or 1. But this holds by a simple perturbation argument on any p_{ij}, q_i, or r_j for which this is not true.

So now consider the matching μ defined by

$$(M_i, W_j) \in \mu \iff \bar{p}_{ij} = 1$$

We want to find stable feasible utility levels compatible with μ.

Take the dual of (P2), obtaining (D2):

$$\min \ \left[\sum_{i=1}^{m} R_i + \sum_{j=1}^{n} S_j\right] + \sum_{i=1}^{m} u_i + \sum_{j=1}^{n} v_j$$

(3.8) s.t. $u_i + v_j \geq c_{ij} - R_i - S_j$

(3.9) $u_i, v_j \geq 0$.

Let $\{\bar{u}_i\}$ and $\{\bar{v}_j\}$ solve (D2). Then consider the following interpretation:

1) $\bar{u}_i$ is the amount of utility man i receives <u>in excess of his reservation utility</u> R_i.
2) $\bar{v}_j$ is the amount of utility woman j receives <u>in excess of her reservation utility</u> S_j

<u>Claim</u>: The optimal dual variables $\{\bar{u}_i\}$ and $\{\bar{v}_j\}$ are feasible utility levels under μ.

<u>Proof</u>: First, if man i is matched by μ, say, to woman j,

$$\bar{p}_{ij} = 1 \implies \text{constraint (3.7) is "loose"}$$

(3.10) $$\implies \bar{u}_i + \bar{v}_j = c_{ij} - R_i - S_j$$

by the complementary slackness theorem of linear programming. But, due to the special structure of the utility functions, the pair (M_i, W_j) can attain <u>any</u> "excess" utilities u_i and v_j satisfying (3.10).

Now suppose man i is unmatched by μ. Then constraint (3.5) is loose, and so $\bar{u}_i = 0$ again by complementary slackness. So indeed man i receives his reservation utility R_i.

And, if woman j is unmatched by μ, she receives S_j by a similar argument.

<u>Claim</u>: $\{\bar{u}_i\}$, $\{\bar{v}_j\}$ are stable utility levels.

<u>Proof</u>: This holds because $\{\bar{u}_i\}$, $\{\bar{v}_j\}$ satisfy constraints (3.8) and (3.9).

To prove the converse of the Theorem, we need to show that if $\{\tilde{p}_{ij}\}$ does not solve (P1) or (P2), then it cannot support "excess utilities" satisfying (3.8) and (3.9). However, note that if $\{u_i\}$ and $\{v_j\}$ are given the interpretation as excess utilities in (D2), then the objective function in that program is $\sum$ (reservation utilities) + $\sum$ (excess utilities), i.e., the global utility. And, by the weak duality theorem,

$$\begin{array}{c} \text{global utility} \\ \text{implied by} \\ \{u_i\}, \{v_j\} \\ \text{satisfying} \\ (3.8), (3.9) \end{array} \geq \begin{array}{c} \text{maximum} \\ \text{global utility} \\ \text{attainable in} \\ (P1) \text{ or } (P2) \\ [\text{by } \{\bar{p}_{ij}\}] \end{array} > \begin{array}{c} \text{global utility} \\ \text{attained} \\ \text{by } \{\tilde{p}_{ij}\} \end{array} .$$

<u>Remark</u>: Thus we can find the set of core matchings simply by using the simplex method on program (P1) or (P2). To show how to find the man-optimal utilities, we first restate Lemma 2.1:

<u>Lemma 3.2</u>: The set of men's utilities $\{u_i\}$ for which there exist utilities $\{v_j\}$ satisfying

$$(\{u_i\}, \{v_j\}) \text{ is in the core}$$

is a lattice.

So, to find the man-optimal utilities, just let Z be the optimal objective value of (P1), and solve program (D3):

$$\max \sum_{i=1}^{m} u_i$$

$$(3.11) \quad \text{s.t.} \quad \sum_{i=1}^{m} u_i + \sum_{j=1}^{n} v_j + \sum_{i=1}^{m} R_i + \sum_{j=1}^{n} S_j = Z$$

$$(3.12) \quad u_i + v_j \geq c_{ij} - R_i - S_j$$

$$(3.13) \quad u_i, v_j \geq 0 .$$

<u>Remark</u>: The above analysis holds for any utility functions $u_{ij}(\alpha)$ and $v_{ij}(\alpha)$ as long as there exist constants c_{ij} satisfying

$$u_{ij}(\alpha) + v_{ij}(-\alpha) = c_{ij}$$

for all α, i, and j.

4. The Linear Nonseparable Case

In this section we drop the assumption of separability, i.e., a person's utility for money depends on with whom he/she is matched. Thus, the utility functions are represented as:

$$u_{ij}(\alpha) = u_{ij} + a_{ij}\alpha \quad \text{and} \quad v_{ij}(\alpha) = v_{ij} + b_{ij}\alpha$$

where u_{ij}, v_{ij}, a_{ij}, and b_{ij} are constants. The inverse functions are:

$$f_{ij}(u_i) = \frac{u_i - u_{ij}}{a_{ij}} \quad , \qquad g_{ij}(v_j) = \frac{v_j - v_{ij}}{b_{ij}}$$

Now consider any matching μ, and let α_i be the amount that man i receives from his mate $\mu[i]$. (α_i does not exist if i isn't matched by μ.) Then the feasibility and stability constraints (2.1) - (2.4) become:

(4.1) $\quad u_{i\mu[i]} + a_{i\mu[i]}\alpha_i \geq R_i$ (for i matched)

(4.2) $\quad v_{\mu[j]j} - b_{\mu[j]j}\alpha_{\mu[j]} \geq S_j$ (for j matched)

(4.3) $\quad \dfrac{R_i - u_{ij}}{a_{ij}} + \dfrac{S_j - v_{ij}}{b_{ij}} \geq 0$ (neither i nor j matched)

(4.4) $\quad \dfrac{u_{i\mu[i]} + a_{i\mu[i]}\alpha_i - u_{ij}}{a_{ij}} + \dfrac{S_j - v_{ij}}{b_{ij}} \geq 0$ (for i matched, j unmatched)

(4.5) $\quad \dfrac{R_i - u_{ij}}{a_{ij}} + \dfrac{v_{\mu[j]j} - b_{\mu[j]j}\alpha_{\mu[j]} - v_{ij}}{b_{ij}} \geq 0$ (for j matched, i unmatched)

$$(4.6)\quad \frac{u_{i\mu[i]} + a_{i\mu[i]}\alpha_i - u_{ij}}{a_{ij}} + \frac{v_{\mu[j]j} - b_{\mu[j]j}\alpha_{\mu[j]} - v_{ij}}{b_{ij}} \geq 0 \qquad \text{(for } i \text{ matched, } j \text{ matched)}$$

Next, define

$$(4.7)\quad \bar{u}_{ij} = \begin{cases} u_{ij} + a_{ij}\alpha_i - R_i & \text{if } i \text{ matches } j \\ 0 & \text{otherwise} \end{cases}$$

$$(4.8)\quad \bar{v}_{ij} = \begin{cases} v_{ij} - b_{ij}\alpha_i - S_j & \text{if } i \text{ matches } j \\ 0 & \text{otherwise} \end{cases}$$

$$(4.9)\quad x_{ij} = \begin{cases} 1 & \text{if } i \text{ matches } j \\ 0 & \text{otherwise} \end{cases}$$

We interpret $\bar{u}_{ij}$ as the excess utility that man i derives from being matched to woman j. A similar interpretation holds for $\bar{v}_{ij}$. Conditions (4.1) - (4.6) become

$$(4.10)\quad \bar{u}_{ij}(1 - x_{ij}) = 0$$

$$(4.11)\quad \bar{v}_{ij}(1 - x_{ij}) = 0$$

$$(4.12)\quad x_{ij}(\bar{u}_{ij} - a_{ij}\alpha_i + R_i - u_{ij}) = 0$$

$$(4.13)\quad x_{ij}(\bar{v}_{ij} + b_{ij}\alpha_i + S_j - v_{ij}) = 0$$

$$(4.14)\quad \sum_{j=1}^{n} x_{ij} \leq 1$$

(4.15) $$\sum_{i=1}^{m} x_{ij} \leq 1$$

(4.16) $$\frac{R_i + [\sum_{\ell=1}^{n} \bar{u}_{i\ell}] - u_{ij}}{a_{ij}} + \frac{S_j + [\sum_{k=1}^{m} \bar{v}_{kj}] - v_{ij}}{b_{ij}} \geq 0$$

(4.17) $\bar{u}_{ij}, \bar{v}_{ij} \geq 0$, α_i unrestricted, $x_{ij} = 0$ or 1 for all i, j.

A few words of explanation may be in order here. First of all, equations (4.10) - (4.13) are the definitions of $\bar{u}_{ij}$ and $\bar{v}_{ij}$ [see (4.7), (4.8)]. Next, conditions (4.14) - (4.15) arise from the definition of matching. Finally, constraints (4.16) are equivalent to conditions (4.3) - (4.6), while (4.1) and (4.2) are reflected by the nonnegativity of the variables $\{\bar{u}_{ij}\}$ and $\{\bar{v}_{ij}\}$.

So, denote by "System 1" the set of constraints (4.10) - (4.17).

Next, consider the following system of equations, which we call "System 2":

(4.18) $$\bar{u}_{ij}(1 - x_{ij}) = \bar{v}_{ij}(1 - x_{ij}) = s_{ij}x_{ij} = 0$$

(4.19) $$\sum_{j=1}^{n} x_{ij} \leq 1$$

(4.20) $$\sum_{i=1}^{m} x_{ij} \leq 1$$

$$(4.21) \qquad \frac{R_i + [\sum_{\ell=1}^{n} \bar{u}_{i\ell}] - u_{ij}}{a_{ij}} + \frac{S_j + [\sum_{k=1}^{m} \bar{v}_{kj}] - v_{ij}}{b_{ij}} - s_{ij} = 0$$

$$(4.22) \qquad \bar{u}_{ij},\ \bar{v}_{ij},\ s_{ij} \geq 0, \quad x_{ij} = 0 \text{ or } 1 \quad \text{for all } i, j.$$

<u>Proposition</u> <u>4.1</u>: System 1 and System 2 are equivalent.

<u>Proof</u>: Suppose $(\{\bar{u}_{ij}\}, \{\bar{v}_{ij}\}, \{x_{ij}\}, \{\alpha_i\})$ solves System 1. Then, setting s_{ij} equal to the amount of slack in constraints (4.16), we claim $(\{\bar{u}_{ij}, \bar{v}_{ij}, x_{ij}, s_{ij}\})$ solves System 2. This is obvious, except for the condition that $s_{ij}x_{ij} = 0$. So suppose $x_{\tilde{i}\tilde{j}} = 1$. Then by (4.10), (4.11), (4.14), and (4.15), $\bar{u}_{\tilde{i}j} = \bar{v}_{i\tilde{j}} = 0$ for $i \neq \tilde{i}$ and $j \neq \tilde{j}$. Substituting (4.12) and (4.13) into (4.16) indeed implies $s_{\tilde{i}\tilde{j}} = 0$.

Now let $(\{\bar{u}_{ij}, \bar{v}_{ij}, x_{ij}, s_{ij}\})$ solve System 2. We aim to show that $(\{\bar{u}_{ij}, \bar{v}_{ij}, x_{ij}\})$ is a solution to System 1. The only difficulty here is to show we can find $\{\alpha_i\}$ such that constraints (4.12) and (4.13) hold. So again suppose $x_{\tilde{i}\tilde{j}} = 1$. This time constraints (4.18) - (4.20) together imply that $\bar{u}_{\tilde{i}j} = \bar{v}_{i\tilde{j}} = 0$ for $i \neq \tilde{i}$ and $j \neq \tilde{j}$. Thus, by (4.21),

$$\frac{R_{\tilde{i}} + \bar{u}_{\tilde{i}\tilde{j}} + \bar{v}_{\tilde{i}\tilde{j}}}{a_{\tilde{i}\tilde{j}}} + \frac{S_{\tilde{j}} + \bar{u}_{\tilde{i}\tilde{j}} + \bar{v}_{\tilde{i}\tilde{j}}}{b_{\tilde{i}\tilde{j}}} = 0 \, .$$

But this implies we can find $\alpha_{\tilde{i}}$ such that, for $i = \tilde{i}$ and $j = \tilde{j}$, (4.12) and (4.13) hold.

So we seek an algorithm to solve System 2. Algebraically, we do this by first solving a relaxation of the problem, and then gradually perturbing it until System 2 is solved. Qualitatively, we accomplish this by using a generalization of the Gale-Shapley algorithm.

First, define the constants

$$\gamma_{ij} = \frac{R_i - u_{ij}}{a_{ij}} + \frac{S_j - v_{ij}}{b_{ij}}$$

So, we can rewrite (4.21) as:

$$(4.21') \qquad \gamma_{ij} + \frac{\sum_{\ell=1}^{n} \bar{u}_{i\ell}}{a_{ij}} + \frac{\sum_{k=1}^{m} \bar{v}_{kj}}{b_{ij}} - s_{ij} = 0$$

Next, consider the following changes to System 2:

1) Remove constraint (4.20)
2) Replace (4.21') with (4.21"):

$$(4.21'') \qquad \gamma_{ij} + \frac{\sum_{\ell=1}^{n} \bar{u}_{i\ell}}{a_{ij}} + \frac{\max_{k=1,\ldots,m} \bar{v}_{kj}}{b_{ij}} - s_{ij} = 0$$

Denote this newly defined system as System 2R.

<u>Remark</u>: The constraints comprising System 2R entail all the requirements for a stable feasible matching except that women are allowed to retain more than one man. A woman with more than one man

evaluates her utility from all of her suitors, and attains the maximum such value.

<u>Lemma 4.2</u>: Any solution to System 2R which also satisfies (4.20) is a solution to System 2.

<u>Definition</u>: Given the constants $\{R_i\}$, $\{S_j\}$, $\{u_{ij}\}$, $\{v_{ij}\}$, $\{a_{ij}\}$, and $\{b_{ij}\}$, define the <u>U-Solution</u> in the following way:

1) Set $\beta_i = \min_j a_{ij}\gamma_{ij}$ for every i

Let $j^*(i)$ be the argmin if $\beta_i < 0$; be undefined if $\beta_i \geq 0$

2) Let $x_{ij} = 1$, $\bar{u}_{ij} = -\beta_i$ if $j = j^*(i)$

Let $x_{ij} = 0$, $\bar{u}_{ij} = 0$ if $j \neq j^*(i)$

Let $\bar{v}_{ij} = 0$ for every i, j.

3) Finally, given $\{\bar{u}_{ij}\}$ defined above, let

$$s_{ij} = \gamma_{ij} + \frac{\sum_{\ell=1}^{n} \bar{u}_{i\ell}}{a_{ij}} \quad \text{for all } i, j.$$

<u>Remark</u>: The U-Solution solves System 2R.

Remark: Suppose each man asks himself, "If I give her her reservation utility, which woman will make me the happiest?". In fact, the U-Solution is the result if this assignment-utility allocation is carried out. It is stable because no man would want to leave such an arrangement. However, there is a problem in that some women are likely to be matched with more than one man. Hence, the U-Solution solves System 2R but in general not System 2.

Next, we describe what will become the iterative step of our procedure.

Lemma 4.3: Let $\mathbf{z} = \{x_{ij}, \bar{u}_{ij}, \bar{v}_{ij}, s_{ij}\}$ solve System 2R, but not System 2 [i.e., constraint (4.20) doesn't hold]. Thus, there exists a $\bar{j}$ for which the set

$$K(\bar{j}) = \{i: \ x_{i\bar{j}} = 1\}$$

contains at least two elements.

Now perturb $\mathbf{z}$ in the following way:

1) Let $v_j = \max_{i=1,\ldots,m} \bar{v}_{ij}$ for every j.

2) Let $u_i = \max_{j \neq \mu[i]} \left(a_{ij} \left[-\gamma_{ij} - \frac{v_j}{b_{ij}} \right]^+ \right)$ for each i.

$$\text{Let } j^*(i) \begin{cases} \text{be the argmax} & \text{if } u_i > 0 \\ \text{not exist} & \text{if } u_i = 0 . \end{cases}$$

3) Let $v^* = \max_{i \in K(\bar{j})} \left(b_{i\bar{j}} \left[-\gamma_{i\bar{j}} - \frac{u_i}{a_{i\bar{j}}} \right]^+ \right)$

Let i^* be the argmax.

4) Set $x_{i^*\bar{j}} = 1$, $\bar{v}_{i^*\bar{j}} = v^*$, $\bar{u}_{i^*\bar{j}} = u_{i^*}$,

$x_{i^*j} = x_{i\bar{j}} = 0$ for all $i \neq i^*$, $j \neq \bar{j}$.

5) For $i \in K(\bar{j})$ but $i \neq i^*$, set:

$x_{ij^*(i)} = 1$, $\bar{u}_{ij^*(i)} = u_i$, $\bar{v}_{ij^*(i)} = v_{j^*(i)}$,

$x_{ij} = 0$ for all $j \neq j^*(i)$.

Note that if $j^*(i)$ does not exist, i becomes unmatched.

6) For $i \notin K(\bar{j})$, do not change x_{ij}, $\bar{u}_{ij}$, or $\bar{v}_{ij}$ for any j.

7) With the values for $\{x_{ij}\}$, $\{\bar{u}_{ij}\}$, and $\{\bar{v}_{ij}\}$ given by 4) - 6), define updated values for $\{s_{ij}\}$ using (4.21").

Call the end result $\mathbf{z}'$. Then $\mathbf{z}'$ solves System 2R also.

<u>Remark</u>: Again, it is helpful to understand qualitatively what is going on here. First, for each woman j, Step 1) formally defines v_j as the utility of the best offer(s) so far received. Next, suppose some woman $\bar{j}$ has at least two offers in hand $\left[|K(\bar{j})| \geq 2 \right]$. When this occurs, the associated men "bid" for her as follows.

First, each such suitor i evaluates his "next best alternative" by considering the set of all <u>other</u> women. With the stipulation that any woman j must be provided with at least v_j, he determines the one $[j^*(i)]$ with whom he could attain the highest utility. This utility is denoted u_i [Step 2) above].

In general, u_i is lower than the utility $\bar{u}^z_{i\bar{j}}$ he now derives from his relationship with $\bar{j}$ [See Claim below]. Thus, by proposing to lower his utility to u_i, he offers to increase $\bar{j}$'s utility if only she would stay with him.

In Step 3), woman $\bar{j}$ makes her choice from amongst these offers, the lucky suitor being i^*. All of the others are rejected, and now make offers to their "next best alternatives" [Steps 4), 5) above--of course, if $u_i = 0$, the "next best alternative" for man i is to attain an excess utility of 0 by remaining unmatched].

Before proving Lemma 4.3, we must first state and prove a Proposition.

<u>Proposition 4.4</u>: Let $\underline{v}^z = \{v^z_j\}$ be the vector of women's utilities defined in Step 1) for z. Then $\underline{v}^z \leq \underline{v}^{z'}$.

<u>Proof</u>: We begin with a Claim:

<u>Claim</u>: If $x^z_{ij} = 1$, then $u_i \leq \bar{u}^z_{ij}$.

<u>Proof</u>: Suppose not. Then,

$$(a_{ij^*}[-\gamma_{ij^*} - \frac{v_{j^*}}{b_{ij^*}}]) > \bar{u}^z_{ij} ,$$

where j^* is defined as in 2) above.

But this implies

$$\gamma_{ij^*} + \frac{v_{j^*}}{b_{ij^*}} + \frac{\bar{u}^{z}_{ij}}{a_{ij^*}} < 0 ,$$

which contradicts (4.21") for z.

To show the Proposition, note that from the above definitions, it is clear that as we pass from z to z', the only component of $\underline{v}$ which changes is $v_{\bar{j}}$. But,

$$v^{z}_{\bar{j}} = \bar{v}^{z}_{\hat{i}\bar{j}} = b_{\hat{i}\bar{j}}\left[-\gamma_{\hat{i}\bar{j}} - \frac{\bar{u}^{z}_{\hat{i}\bar{j}}}{a_{\hat{i}\bar{j}}}\right] \qquad \text{for some } \hat{i} \in K(\bar{j})$$

$$\leq b_{\hat{i}\bar{j}}\left[-\gamma_{\hat{i}\bar{j}} - \frac{u_{\hat{i}}}{a_{\hat{i}\bar{j}}}\right] \qquad \text{(by the Claim)}$$

$$\leq v^{*} = v^{z'}_{\bar{j}} \qquad \text{by def'n of } v^{*} .$$

<u>Proof of Lemma 4.3</u>: The only difficult things to prove are $s^{z'}_{ij} \geq 0$ and $s^{z'}_{ij}x^{z'}_{ij} = 0$. Equivalently, we must show that the expression

$$\gamma_{ij} + \frac{\sum_{\ell=1}^{n} \bar{u}^{z'}_{i\ell}}{a_{ij}} + \frac{\max_{k=1,\ldots,m} \bar{v}^{z'}_{kj}}{b_{ij}}$$

is nonnegative for all i and j, and equal to zero if $x^{z'}_{ij} = 1$.

<u>Case 1</u>: $i = i^*$, $j = \bar{j}$. (So $x_{ij}^{z'} = 1$)

Then,

$$\gamma_{ij} + \frac{\sum_{\ell=1}^{n} \bar{u}_{i\ell}^{z'}}{a_{ij}} + \frac{\max_{k=1,\ldots,m} \bar{v}_{kj}^{z'}}{b_{ij}}$$

$$= \gamma_{i^*\bar{j}} + \frac{u_i^*}{a_{i^*\bar{j}}} + \frac{v^*}{b_{i^*\bar{j}}}$$

$$= 0 \qquad \text{by definition of } u_i^*,\ v^* .$$

<u>Case 2</u>: $i = i^*$, $j \neq \bar{j}$.

Now,

$$\gamma_{ij} + \frac{\sum_{\ell=1}^{n} \bar{u}_{i\ell}^{z'}}{a_{ij}} + \frac{\max_{k=1,\ldots,m} \bar{v}_{kj}^{z'}}{b_{ij}}$$

$$= \gamma_{i^*j} + \frac{\max_{\ell \neq \bar{j}} \left(a_{i^*\ell} \left[-\gamma_{i^*\ell} - \frac{v_\ell}{b_{i^*\ell}} \right]^+ \right)}{a_{i^*j}} + \frac{v_j}{b_{i^*j}}$$

$$\geq \gamma_{i^*j} + \frac{a_{i^*j} \left[-\gamma_{i^*j} - \frac{v_j}{b_{i^*j}} \right]}{a_{i^*j}} + \frac{v_j}{b_{i^*j}} = 0 .$$

<u>Case 3</u>: $i \in K(\bar{j})$ but $i \neq i^*$, $j = \bar{j}$.

Then,

$$\gamma_{ij} + \frac{\sum_{\ell=1}^{n} \bar{u}_{i\ell}^{z'}}{a_{ij}} + \frac{\max_{k=1,\ldots,m} \bar{v}_{kj}^{z'}}{b_{ij}}$$

$$= \gamma_{i\bar{j}} + \frac{u_i}{a_{i\bar{j}}} + \frac{v^*}{b_{i\bar{j}}}$$

$$\geq \gamma_{i\bar{j}} + \frac{u_i}{a_{i\bar{j}}} + \frac{b_{i\bar{j}}\left[-\gamma_{i\bar{j}} - \frac{u_i}{a_{i\bar{j}}}\right]^+}{b_{i\bar{j}}} \geq 0 \ .$$

<u>Case 4</u>: $i \in K(\bar{j})$ but $i \neq i^*$, $j = j^*(i)$ [$\neq \bar{j}$] .

Here again, $x_{ij}^{z'} = 1$. And, since $j^*(i)$ exists,

$$0 < u_i = a_{ij^*(i)}\left[-\gamma_{ij^*(i)} - \frac{v_{j^*(i)}}{b_{ij^*(i)}}\right] .$$

Thus,

$$\gamma_{ij} + \frac{\sum_{\ell=1}^{n} \bar{u}_{i\ell}^{z'}}{a_{ij}} + \frac{\max_{k=1,\ldots,m} \bar{v}_{kj}^{z'}}{b_{ij}}$$

$$= \gamma_{ij^*(i)} + \frac{u_i}{a_{ij^*(i)}} + \frac{v_{j^*(i)}}{b_{ij^*(i)}}$$

$$= 0 \qquad \text{by definitions of } u_i,\ j^*(i) \ .$$

<u>Case 5</u>: $i \in K(\bar{j})$ but $i \neq i^*$, $j \neq j^*(i)$, $j \neq \bar{j}$.

Now,

$$\gamma_{ij} + \frac{\sum_{\ell=1}^{n} \bar{u}_{i\ell}^{z'}}{a_{ij}} + \frac{\max_{k=1,\ldots,m} \bar{v}_{kj}^{z'}}{b_{ij}}$$

$$= \gamma_{ij} + \frac{u_i}{a_{ij}} + \frac{v_j}{b_{ij}}$$

$$\geq \gamma_{ij} + \frac{a_{ij}\left[-\gamma_{ij} - \frac{v_j}{b_{ij}}\right]^+}{a_{ij}} + \frac{v_j}{b_{ij}} \geq 0 \ .$$

Case 6: $i \notin K(\bar{j}),\ \ j = \bar{j}$.

We have

$$\gamma_{ij} + \frac{\sum_{\ell=1}^{n} \bar{u}_{i\ell}^{z'}}{a_{ij}} + \frac{\max_{k=1,\ldots,m} \bar{v}_{kj}^{z'}}{b_{ij}}$$

$$= \gamma_{i\bar{j}} + \frac{\sum_{\ell=1}^{n} \bar{u}_{i\ell}^{z}}{a_{i\bar{j}}} + \frac{v^*}{b_{i\bar{j}}} \qquad \text{by 6) above .}$$

$$\geq \gamma_{i\bar{j}} + \frac{\sum_{\ell=1}^{n} \bar{u}_{i\ell}^{z}}{a_{i\bar{j}}} + \frac{v_{\bar{j}}^{z}}{b_{i\bar{j}}} \qquad \text{by Proposition 4.4 .}$$

$$\geq 0 \qquad \text{because } z \text{ satisfies System 2R .}$$

Case 7: $i \notin K(\bar{j}),\ \ j \neq \bar{j}$.

In this case, none of the quantities

$$x_{ij}\ ,\ \sum_{j=1}^{n} \bar{u}_{ij}\ ,\ \text{or}\ \max_{i=1,\ldots,m} \bar{v}_{ij}$$

change as we move from z to z'. Hence, $s_{ij}^{z} \geq 0$ and $s_{ij}^{z} x_{ij}^{z} = 0$ imply $s_{ij}^{z'} \geq 0$ and $s_{ij}^{z'} x_{ij}^{z'} = 0$.

Thus, we have the makings of an algorithm. We start with the U-Solution, which solves System 2R. If it also satisfies (4.20), it solves System 2 [Lemma 4.2], and thus is in the core. If not, we perturb it in the manner described above [Lemma 4.3], obtaining another solution to System 2R with a higher $\underline{v} = \{v_j\}$. Again we check condition (4.20), and so on.

However, we need to show that the algorithm will terminate in a finite number of steps. We now give a nondegeneracy assumption which insures that this occurs.

But first, a necessary preliminary:

Proposition 4.5: Consider the algorithm just described, and let $z = \{\bar{u}^z_{ij}, \bar{v}^z_{ij}, x^z_{ij}, s^z_{ij}\}$ be a solution obtained at any step. Then, for every j,

$$v^z_j \equiv \max_{i=1,\ldots,m} \bar{v}^z_{ij}$$

is equal to $\bar{v}^z_{ij}$ for any i in $K(j)$.

Proof: *This property holds trivially for the* U-Solution because $\bar{v}^{U\text{-}Sol}_{ij} = 0$ for every i and j. And, it is also easy to see that the "pivoting" procedure defined above preserves it.

Theorem 4.6: Suppose there exists an $\varepsilon > 0$ such that, for all but a finite number of iterations,

$$v^* = b_{i^*\bar{j}}\left[-\gamma_{i^*\bar{j}} - \frac{u_{i^*}}{a_{i^*\bar{j}}}\right]^+$$

(4.23)

$$\geq \max_{i\in K(\bar{j}),i\neq i^*} \left(b_{i\bar{j}}\left[-\gamma_{i\bar{j}} - \frac{u_i}{a_{i\bar{j}}}\right]^+\right) + \varepsilon .$$

Then the algorithm terminates at a stable solution after a finite number of iterations.

In order to prove this, we again need a preliminary result:

<u>Claim</u>: Suppose we "pivot" on a $\bar{j}$ such that (4.23) holds. Then $v_{\bar{j}}^{z'} \geq v_{\bar{j}}^{z} + \varepsilon$.

<u>Proof</u>: By Proposition 4.5,

$$v_{\bar{j}}^{z} = v_{i\bar{j}}^{z} = b_{i\bar{j}}\left[-\gamma_{i\bar{j}} - \frac{u_{i\bar{j}}^{z}}{a_{i\bar{j}}}\right] \qquad \text{for } \underline{\text{any}} \quad i \in K(\bar{j}) .$$

In particular, choose $i = \tilde{i} \neq i^*$, with $\tilde{i} \in K(\bar{j})$. Then,

$$v_{\bar{j}}^{z} = b_{\tilde{i}\bar{j}}\left[-\gamma_{\tilde{i}\bar{j}} - \frac{u_{\tilde{i}\bar{j}}^{z}}{a_{\tilde{i}\bar{j}}}\right]$$

$$\leq b_{\tilde{i}\bar{j}}\left[-\gamma_{\tilde{i}\bar{j}} - \frac{u_{\tilde{i}}}{a_{\tilde{i}\bar{j}}}\right] \qquad \text{by the Claim in the proof of Proposition 4.4}$$

$$\leq v_{\bar{j}}^{z'} - \varepsilon \qquad \text{by (4.23)} .$$

Thus, condition (4.23) implies that in an infinite number of iterations, <u>some</u> woman's utility increases by ε. So, Theorem 4.6 will follow because the set of core utilities for women is bounded.

Formally,

<u>Proof of Theorem 4.6</u>: Suppose the algorithm does not terminate, and consider z after

$$m + \sum_{j=1}^{n} \left\lceil \frac{\max_i \, [-\gamma_{ij} b_{ij}]^+}{\varepsilon} \right\rceil$$

pivots in which (4.23) is satisfied. Then, because of the preceding Claim, one can see that in at least m of these maneuvers, we are pivoting on a $\bar{j}$ for which

$$v_{\bar{j}} \geq \max_i \; -\gamma_{i\bar{j}} b_{i\bar{j}} \; . \tag{4.24}$$

But (4.24) implies that

$$\bar{u}_{i\bar{j}} = a_{i\bar{j}} \left[-\gamma_{i\bar{j}} - \frac{v_{\bar{j}}}{b_{i\bar{j}}} \right] \leq 0$$

for all $i \in K(\bar{j})$, which in turn implies $u_i = 0$ for $i \neq i^* \in K(\bar{j})$. Thus, each such pivot removes a man from the matching [Step 5) of the algorithm]. So, z must have all m men removed from the matching, and so trivially satisfies (4.20).

<u>Remark</u>: By a similar argument, we can show that the conclusion of the Theorem remains valid if we replace condition (4.23) with

$$\bar{u}_{\tilde{i}\bar{j}} \geq u_{\tilde{i}} + \varepsilon \quad \text{for some } \tilde{i} \neq i^* \in K(\bar{j}) \; . \tag{4.25}$$

Remark: We reiterate that this algorithm looks like the analogue of the Gale-Shapley algorithm, which treats the simpler, "preference ordering" case. This is because:

1) The algorithm starts with each man proposing to the woman who will give him the most utility [i.e., the U-Solution].

2) The algorithm continues so long as some woman $\bar{j}$ has two or more proposals out of which she accepts one and rejects the rest [i.e., $|K(\bar{j})| \geq 2$. i^* is the lucky man.]

3) If a man gets rejected, he proposes next to another woman who can give him the most utility [Steps 2), 5) of the algorithm.]

4) As the algorithm continues, the women's position improves [Proposition 4.4] and the men's gets worse.

5) The algorithm terminates with a man-optimal matching, although not necessarily with the man-optimal utilities. The proof of this is in a forthcoming paper (Quint 1985).

5. The Nonlinear Separable and Nonlinear Nonseparable Cases

In this section, we show how the algorithm defined in Section 4 generalizes to cover the case where $u_{ij}(\alpha)$ and $v_{ij}(\alpha)$ are arbitrary strictly increasing functions. All of the results of the previous section hold in this more general setting.

Recall from Section 2 the definitions

$u_{ij}(\alpha)$ = utility to man i of being matched to woman j and receiving payment of α

$v_{ij}(\alpha)$ = utility to woman j of being matched to man i and receiving payment of α

$f_{ij}(u_i)$ = inverse function of $u_{ij}(\alpha)$

$g_{ij}(v_j)$ = inverse function of $v_{ij}(\alpha)$

R_i = reservation utility of man i

S_j = reservation utility of woman j

Again consider any matching μ, and let α_i be the amount that man i receives from his mate $\mu[i]$. Define

$$(5.1) \qquad \bar{u}_{ij} = \begin{cases} u_{ij}(\alpha_i) - R_i & \text{if } i \text{ matches } j \\ 0 & \text{otherwise} \end{cases}$$

$$(5.2) \qquad \bar{v}_{ij} = \begin{cases} v_{ij}(-\alpha_i) - S_j & \text{if } i \text{ matches } j \\ 0 & \text{otherwise} \end{cases}$$

$$(5.3) \qquad x_{ij} = \begin{cases} 1 & \text{if } i \text{ matches } j \\ 0 & \text{otherwise} \end{cases}$$

Proceeding as in the previous section, we can again pose the problem in terms of a "System 2":

$$(5.4) \qquad \bar{u}_{ij}(1 - x_{ij}) = \bar{v}_{ij}(1 - x_{ij}) = s_{ij}x_{ij} = 0$$

$$(5.5) \qquad \sum_{j=1}^{n} x_{ij} \leq 1$$

$$(5.6) \qquad \sum_{i=1}^{m} x_{ij} \leq 1$$

$$(5.7) \qquad f_{ij}\left(R_i + \sum_{\ell=1}^{n} \bar{u}_{i\ell} \right) + g_{ij}\left(S_j + \sum_{k=1}^{m} \bar{v}_{kj} \right) - s_{ij} = 0$$

$$(5.8) \qquad \bar{u}_{ij},\ \bar{v}_{ij},\ s_{ij} \geq 0, \quad x_{ij} = 0 \text{ or } 1.$$

The next step is to define the relaxation "System 2R". We do this by making the following changes to (5.4) - (5.8):

1) Remove constraint (5.6)

2) Replace (5.7) with (5.7"):

$$(5.7") \qquad f_{ij}\left(R_i + \sum_{\ell=1}^{n} \bar{u}_{i\ell} \right) + g_{ij}\left(S_j + \max_{k=1,\dots,m} \bar{v}_{kj} \right) - s_{ij} = 0$$

<u>Remark</u>: Any solution to System 2R which also satisfies (5.6) solves System 2.

<u>Definition</u>: Given the functions $u_{ij}(\alpha)$, $v_{ij}(\alpha)$ and the constants $\{R_i\}$, $\{S_j\}$, define the <u>U-Solution</u> in the following way:

1) Set $\beta_i = \max_j u_{ij}(-g_{ij}(S_j)) - R_i$ for every i.

$$\text{Let } j^*(i) \quad \begin{cases} \text{be the argmax if } \beta_i > 0 \\ \text{be undefined if } \beta_i \leq 0 \end{cases}$$

2) Let $\begin{array}{l} x_{ij} = 1 \\ \bar{u}_{ij} = \beta_i \end{array}$ if $j = j^*(i)$

Let $\begin{array}{l} x_{ij} = 0 \\ \bar{u}_{ij} = 0 \end{array}$ if $j \neq j^*(i)$

Let $\bar{v}_{ij} = 0$ for every i, j.

3) Finally, given $\{\bar{u}_{ij}\}$ defined above, let

$$s_{ij} = g_{ij}(S_j) + f_{ij}\left(\sum_{\ell=1}^{n} \bar{u}_{i\ell} + R_i \right)$$

<u>Remark</u>: The U-Solution solves System 2R.

The algorithm itself, as defined in the statement of Lemma 4.3, will be unchanged, except that now:

$$(5.9) \qquad u_i = \max_{j \neq \mu[i]} \left[u_{ij}(-g_{ij}(S_j + v_j)) - R_i \right]^+ \quad \text{for each } i.$$

$$(5.10) \qquad v^* = \max_{i \in K(\bar{j})} \left[v_{i\bar{j}}(-f_{i\bar{j}}(R_i + u_i)) - S_{\bar{j}} \right]^+$$

Also note that we now update $\{s_{ij}\}$ using (5.7").

Finally, the analogue of Theorem 4.6 is

<u>Theorem 5.1</u>: Suppose there exists an $\varepsilon > 0$ such that, for all but a finite number of iterations,

$$v^* = [\, v_{i^*\bar{j}}(-f_{i^*\bar{j}}(\, u_{i^*} + R_{i^*})) - S_{\bar{j}} \,]^+$$

$$\geq \max_{i \in K(\bar{j}),\, i \neq i^*} [\, v_{i\bar{j}}(-f_{i\bar{j}}(\, u_i + R_i)) - S_{\bar{j}} \,]^+ + \varepsilon \,.$$

Then the algorithm terminates at a stable solution after a finite number of iterations.

6. Topics for Further Research

The results in this paper raise a number of important issues. One is the question of how strong a condition (4.24) is, or, whether it is even necessary at all for convergence. All of the sample problems run so far have both converged and satisfied this constraint, but testing has been limited and more work needs to be done.

Next, it might be interesting to investigate the complexity of the algorithm, possibly over various utility functions and/or pivoting rules (i.e., rules for choosing $\bar{j}$ when $|K(j)| \geq 2$ for more than one j).

Finally, one could try to apply this work to various real-world situations, such as job-matching-salary determination or a "college admissions problem with scholarships allowed" setup.

References

Crawford, V.P., and E.L. Knoer, "Job Matching with Heterogenous Firms and Workers", Econometrica, Volume 49, No. 2, March 1981.

Dantzig, George, Linear Programming and Extensions, Princeton: Princeton University Press, 1963.

Gale, David, The Theory of Linear Economic Models, New York: MacGraw Hill, 1960, Theorem 3.3, p. 84.

Gale, David, and Gabrielle Demange, "The Strategy Structure of Two-Sided Matching Markets", unpublished technical report, August, 1984.

Gale, David, and Lloyd Shapley, "College Admission and the Stability of Marriage", American Mathematical Monthly, Volume 69, pp. 9-14, 1962.

Quint, Thomas, forthcoming dissertation, Department of Operations Research, Stanford University.

Quinzii, Martine, "Core and Competitive Equilibria with Indivisibilities", International Journal of Game Theory, Volume 13, Issue I (1984), pp. 41-60.

Roth, Alvin E., "The Evolution of the Labor Market for Medical Interns and Residents: A Case Study in Game Theory", working paper #165, University of Pittsburgh, 1983.

Scarf, Herbert, "The Core of an N-Person Game", Econometrica, Vol. 35, No. 1, January 1967.

Shapley, L.S., and M. Shubik, "The Assignment Game I: The Core", International Journal of Game Theory, Volume 1, pp. 111-130.

END

FILMED

12-85

DTIC